Science Journal of Education
2014; 2(1): 22-32
Published online March 10, 2014 (http://www.sciencepublishinggroup.com/j/sjedu)
doi: 10.11648/j.sjedu.20140201.14

Methods of solving Diophantine equations in secondary education in Romania

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To cite this article:
Lupu Costică. Methods of Solving Diophantine Equations in Secondary Education in Romania. *Science Journal of Education.* Vol. 2, No. 1, 2014, pp. 22-32. doi: 10.11648/j.sjedu.20140201.14

Abstract: This study aims to highlight the importance of knowing the methods of solving Diophantine equations. The material is structured into: Introduction, Classes of Diophantine equations, presentation of first-degree Diophantine equations, Pythagorean triples and higher - Diophantine equations, methods for solving Diophantine equations. The paper describes and exemplifies different methods such as the decomposition method, the parametric method for solving Diophantine equations, solving Diophantine equations with inequalities through the method of modular arithmetic, mathematical induction, Fermat's method of infinite descent. Solving problems is illustrated by various applications of the mathematical results methods presented above. Any education, including mathematical education, has a double effect. On the one hand, the learner gains knowledge, on the other hand, he builds those skills which are engaged in work, develop the abilities needed to perform this education. Mathematical education builds thought. Of course, other actions are involved in building thought, but the role of Mathematical education is essential. This article is part of an empirical research on the teaching and learning of mathematics, teaching practices related to the main classes of Diophantine equations, leading to the development of cognitive skills in students.

Keywords: Diophantine Equations, Linear Equations, Pythagorean Equations, Bachet Equations

1. Introduction

This paper aims to highlight the important nuances in the study of Diophantine equations. We call Diophantine equation an equation of the form: f (x_1, x_1, ..., x_n) = 0, (1), where f is a function of n variables and n ≥ 2. If f is a polynomial with integer coefficients, the equation is called algebraic Diophantine equation. An n - tuple $(x_1^0, x_2^0, \ldots, x_n^0) \in Z^n$ which satisfies (1) is called a solution of equation (1). An equation that has one or more solvent solutions is called solvable [1].

The problem of finding solutions to equation (1) is completely solved in integers only for equations with one unknown, for equations of the first order and second order equations with two unknowns. It is generally difficult enough even question of the existence the whole solution.

2. Classes of Diophantine Equations

2.1. Diophantine Equations of the first Degree

Definition. A Diophantine equation (1) with two unknown terms: ax + by=c (1) where $a, b, c \in Z, ab \neq 0$. The pair $(x_0, y_0) \in Z^2$ which verifies (1) is called particular solution.

Theorem 1. The necessary and sufficient condition for the equation (1) to admit solutions is d/c, where d=(a, b) [1].

Demonstration. If we have (x_0, y_0) verifying (1), then $ax_0 + by_0$=c and we have d/c, so the condition is necessary. If d/c, there is c1∈ R, so that c=dc_1. When d= (a, b), there is u, v ∈ Z, so that au+bv=d (2). By multiplying the two terms of the inequality (2) with c_1 we get a(c_1u) + b(c_1v)=c, and the pair (c_1u,c_1v) is a particular solution of the equation (1).

Theorem 2. When the Diophantine equation ax+by=c has as particular solution (x_0, y_0) and d=(a, b), the general result is given by: x=x_0 +(b/d)t, y=y_0 -(a/d)t, $t \in Z$ [1].

Demonstration. When (x_0, y_0) is a particular solution, then ax_0+by_0=c (3). For an arbitrary pair (x, y) ∈ Z^2 we have a x +b y =c (4). From the subtractions between the two relations, we get: a(x-x_0)=b(y_0-y) (5). As a=a_1d, b=b_1d, (a_1,b_1) = 1, relation (5) becomes a_1(x-x_0)=b_1(y_0-y) (6). It results that $b_1/a_1(x-x0)$ and $(a_1,b_1) = 1$ we deduce that $b_1/x-x_0$, there is $t \in Z$, so that $x-x_0 = b_1 t$. By replacing in (6) we get $y_0-y=a_1t$. So $x=x_0+b_1t$ end $y=y_0-a_1t$.

Reciprocal, if (x_0, y_0) is a particular solution (verify (3)) and $x=x_0+b_1t$, $y=y_0-a_1t$, $t \in Z$, then (6) implies (5). Finally from (5) and (3) there results (4), so (x, y) is a solution of the equation.

Corollary. Having a_1, a_2 two full primary numbers. If (x_0, y_0) is a solution of the equation $a_1x+a_2y=b$ then all the solutions are given by $x=x_0+a_2t$, $y=y_0-a_1t$ where $t \in Z$.

Example: Find the higher common divisor of the full numbers 1215 and 2755 and express them as a linear combination of the two numbers [3].

Solve in Z the equation 1215 x -2755 y=560.

Solution. a) d=(1215,-2755) = (2755,1215). Applying Euclid algorithm, we have:

$2755=1215\cdot2+325$
$1215=325\cdot3+240$
$325=240\cdot1+85$
$240=85\cdot2+70$
$85=70\cdot1+15$
$70=15\cdot4+10$
$15=10\cdot1+5$
$10=5\cdot2$, so d=5.

To find $u, v \in Z$ so that $d=1215u+(-2755)v$ we use the algorithm above. We have:

$325=1\cdot2755+(-2)\cdot1215$
$240=(-3)\cdot2755+7\cdot1215$
$85=325-240\cdot1=4\cdot2755-9\cdot1215$
$70=240-85\cdot2=(-11)\cdot2755+25\cdot1215$
$15=1\cdot52755-34\cdot1215$
$10=-71\cdot2755+161\cdot1215$
$5=86\cdot2755-195\cdot1215$
So $5=1215\cdot(-195)+(-2755)\cdot(-86)$.

We have 5/560, so the equation has solutions. If $560=5\cdot112$, the particular solution is $x0=-195\cdot112$,

$y_0=-86\cdot112$, and the general solution is $x=x0+551\cdot t$, $y=y_0+243\cdot t$, $t \in Z$.

Definition. A Diophantine equation 1 with n unknown terms ($n \geq 1$) is called Diophantine linear equation and it looks like in the example: $a_1x_1+a_2x_2+\ldots+a_nx_n=b$ (7), where $a_1, a_2, \ldots, a_n$, b are full fixed numbers and $a_1, a_2, \ldots, a_n$, are zero numbers.

Theorem 3. The necessary condition is enough so that equation (7) to have solutions like d/b, where $d= (a_1, a_2, \ldots, a_n)$ [1].

Demonstration. A solution of equation (7) is an ordinate system $(x_1^0, x_2^0, \ldots, x_n^0) \in Z^n$ (8) where there is the equality $a_1x_1+a_2x_2+\ldots+a_nx_n=b$. We presume that b is not divided with d, then the equality (8) is not possible, as the left side of d (8) is divided with d, and the right side can not. If d/b, meaning $b=db_1$, there is $u_1, u_2, \ldots, u_n \in Z$ so that $a_1u_1+a_2u_2+ +\ldots+a_nu_n=d$. Multiplying both terms with b1, we get: $a_1(u_1b_1)+a_2(u_2b_1)+ \ldots+a_n(u_nb_1)=b$ and we emphasized a particular result of the equation (7),

$x_1^0 = u_1b_1, x_2^0 = u_2b_1, \ldots, x_n^0 = u_nb_1$.

In order to find the multitude of solutions for this equation we use the following:

Theorem 4. The solving of the Diophantine equation (1) with n unknown terms reduces to the solving of a Diophantine equation (1) with two unknown terms and a Diophantine equation (1) with n-1 unknown terms. The general solution depends of n-1 full parameters [1].

Demonstration. We have equation (7) which accomplishes the condition d/b, $d=(a_1, a_2, \ldots, a_n)$. We write $d_2=(a_1,a_2)$, $d_3=(d_2, a_3), \ldots, d_k=(d_{k-1},a_k), \ldots, d_n=(d_{n-1},a_n)$. We know that $d=d_n$. We write $a_1x_1+a_2x_2+\ldots+a_{n-1}x_{n-1}=d_{n-1}y$. The equation (7) is equivalently written $d_{n-1}y+a_nx_n= b$ (9). As $(d_{n-1},a_n)=d$ an d/b, equation (9) has solutions and if (y_0, x_n^0) is a particular solution of the equation, the general result is: $y = y_0 + (a_n/d)t$, $x = (d_{n-1}/d)t, t \in Z$. For the fixed value of t, we find a value for x_n and another value for y.

In order to find a solution for equation (7), we need to find, for fixed t, x_1, x_{n-1} so that: $a_1x_1+a_2x_2+\ldots+a_{n-1}x_{n-1}=d_{n-1}(y_0+(a_n/d)t$ (10). If $(a_1,a_2, \ldots,a_{n-1})=d_{n-1}$ and $d_{n-1}/d_{n-1}(y_0 + (a_n/d))$, the equation (10) has solutions.

According to theorem 2, the Diophantine equation 1 with two unknown terms has solutions and the general result depends on a parameter. Supposing that the general result of the equation with n-1 unknown terms depends on the parameters n-2: $t_1, t_2, \ldots, t_{n-2}$. It results that the general solution of equation (7), depends on the n-1 parameters: t_1, $t_2, \ldots, t_{n-2}$ and t. The given demonstration encloses a method of determining the general solution of the equation (7) for $n \geq 2$. We determine x_n according to the parameter t_{n-1}:

$x_n = x_n^0-(d_{n-1}/d)t_{n-1}$, $t_{n-1} \in Z$. From (10) we have x_{n-1} which depends on a new parameter t_{n-2} (and of t_{n-1}) etc., until we get $a_1x_1+a_2x_2=A(t_2, t_3, \ldots, t_{n-1})$, where x_1 and x_2 depend on the t_1 parameter.

Example Solve in Z the Diophantine equation,

$4x_1-6x_2+10x_3+2x_4=14$.

Solution d=(4, -6, 10, 2)=2 and 2/14, so it has solutions like, $d_3=(4, -6, 10)=2$. We write $4x_1-6x_2+10x_3=2y_1$ and the given equation becomes $2y_1+2x_4=14$ or y_1+x_4+7. We write $y_1=t_3$ and we have $x_4=7-t_3$, $t \in Z$. We get $4x_1-6x_2+10x_3=2t_3$ or $2x_1-3x_2+5x_3=t_3$. We write $2x_1-3x_2=y_2$ and we get $y_2+5x_3=t_3$. We write $x_3=t_2$, so $y_2=t_3-5t_2$. From $2x_1-3x_2=t_3-5t_2$, $2(2t_3-10t_2)-3(t_3-5t_2)=t_3-5t_2$, we get $x_1=2t_3-10t_2+3t_1$, $x_2=t_3-5t_2+2t_1$. The general solution is:

$x_1=3t_1-10t_2+2t_3$; $x_2=2t_1-5t_2+t_3$; $x_3=t_2$; $x_4=7-t_3$, $t_1, t_2, t_3 \in Z$.

2.2. The Pythagorean Triples and their Problems

One of the most famous Diophantine equations is called the Pythagoras equation $x^2+y^2=z^2$ (1).

Studied in fine details by Pythagoras in relation with the rectangle triangles whose sides has natural numbers like lengths, this equation has been known since the old times of the Babylon.

Observations: 1. If the triplet (x_0, y_0, z_0) encloses equation (1), then any triplet of the form (kx_0, ky_0, kz_0), $k \in Z$ is also a solution of the equation (1). In order to find all the solutions of the equation (1), consisting in zero numbers, it is enough to find the solutions (x, y, z) for which the numbers are relative primary. The solutions which have relative primary numbers two by two are named primitive solutions.

2. If in a solution (x, y, z) of the equation (1) two of the numbers x, y, z have a then $\lambda \neq \pm 1$ the third number divides with λ.

3. If (x, y, z) is a solution of the equation (1) then (y, x, z) is a solution

4. If (x, y, z) is a solution of equation (1) then x or y is even (if x and y are both uneven then x^2+y^2 would be 4k+2, when the square of a full number can only be 4k or 4k+1).

5. If (x, y, z) is a solution of the equation (1) then ($\pm$ y, $\pm$ x, $\pm$ z) will be solutions.

Theorem Any solution (x, y, z) of natural numbers of equation (1) is $x=2mn$, $y=m^2-n^2$, $z=m^2+n^2$ (2) with m and n natural numbers, primary, different parities and $n < m$ [1].

Demonstration. The entity $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$ shows that the numbers are solutions of the equation (1) with x even. If the numbers x, y, z have a common divisor $\lambda \geq 2$ then λ divides and the numbers $2m^2=(m^2+n^2)^2+(m^2-n^2)^2$ and $2n^2=(m^2+n^2)^2-(m^2-n^2)^2$. When (m, n)=1 it results $\lambda = 2$ but this time m^2 and n^2 are simultaneous even or uneven which is impossible as the hypothetically m and n do not have different parities. The solution is primitive.

Reciprocally, when (x, y, z) is a primitive solution of (1), with x, y, z natural numbers and x an even number (x=2a). Then y and z are uneven, so the numbers z+y and z-y are even (fie z+y =2b and z-y =2c). Any common divisor of b and c divides z=b+c and y=b-c, so $\lambda = \pm 1$, and (b, c)=1. On the other side $4a^2=x^2=z^2-y^2=4bc$, where $a^2=bc$, and $b=m^2$ and $c=n^2$ (m, n$\in$N), and $a^2=m^2n^2 \Leftrightarrow a=mn$, so $x=2a=2mn$, $y=b-c=m^2-n^2$ and $z=b+c=m^2+n^2$ (see n<m).

A triplet (x, y, z) like (2) is called Pythagorean triple.

Corollary All the solutions of equation (1) are given by $x=2rmn$, $y= r(m^2-n^2)$, $z=r(m^2+n^2)$ with r, m, n$\in$Z. The immediate generalization of the equation (1) is given by the equation $x^2+y^2+z^2=t^2$ (3). The positive solutions of equation (3) represents the dimension and length of the diagonal of a rectangle of a parallelepiped. Like in the case of rectangle triangle it is here where intersects where all dimensions are natural numbers.

Theorem all the solutions in natural numbers of the equation (3) are given by x, y, z, t with y, z even numbers and $x = \frac{l^2+m^2-n^2}{n}$, $y=2l$, $z=2m$, $t = \frac{l^2+m^2+n^2}{n}$ (4) where *l*, *m, n* are natural arbitrary numbers, and *n* is a divisor of l^2+m^2 smaller than $\sqrt{l^2+m^2}$. Any solution is obtained once [1].

Demonstration The entity $\left(\frac{l^2+m^2-n^2}{n}\right) + (2l)^2 + (2m)^2 = \left(\frac{l^2+m^2+n^2}{n}\right)^2$ shows that the defined quadruple (4) is a solution of the equation (3) besides, the y numbers, are even.

On the other side, we observe that at least two of the numbers x, y, z are even, so we have $t^2 \equiv 2,3$ (mod4) which is not possible.

Presuming that y=2l, z=2m, where l, m$\in$N*. Writing t-x=u we get $X^2+4l^2+4m^2=(x+u)^2$ and $u^2=4(l^2+m^2)-2ux$. So u^2 is even, and u=2n, where n$\in$N*. It results that $x = \frac{l^2+m^2-n^2}{n}$ and $t=x+u=x+2n=\frac{l^2+m^2+n^2}{n}$, where *l,m,n* $\in$N* and n is a divisor of l^2+m^2 smaller than $\sqrt{l^2+m^2}$. It is easy to see that any solution (x, y, z, t) of equation (3) with y and z even we het one result by using these formulas (4). By (4) we get $l = \frac{y}{2}, m = \frac{z}{2}, n = \frac{t-x}{2}$ and the full numbers re determined by the quadruple (x, y, z, t).

2.3. Superior Diophantine Equations

2.3.1. Bachet Equations

Definition. A Bachet equation looks like $y^2=x^3+k$ with k$\in$ Z*.

Theorem. The equation $y^2=x^3+7$ does not have solutions in Z^2 [3].

Demonstration. If x is even then $y^2 \equiv 3$ (mod 4) which is absurd. Also, if x≡3 (mod 4) then $y^2 \equiv 2$ (mod 4) again absurd. So x≡1(mod 4). Writing $y^2+1=x^3+8=(x+2)(x^2-2x+4)$, and $x^2-2x+4 \equiv 3$(mod 4), we deduce that $y^2 \equiv -1$(mod 4)$\Leftrightarrow$ $y^2 \equiv 3$(mod 4), which is again absurd.

Theorem. The equation $y^2=x^3-16$ does not have solutions in Z^2 [4].

Demonstration. When x is even, x=2a, y is also even, and y=2b with a, b$\in$ Z. Then $b^2+4=2a^3$, with b=2c and a =2d , c, d$\in$ Z. Then $c^2+1=4d^3$ which is absurd. So x and y are uneven. We deduce that $x^3 \equiv 1$(mod 8), so x≡1(mod 8). Then x-2≡-1(mod 8) and $(x-2)/(x^3-8)=y^2+8$. We deduce that x-2 can not have primary factors p like p≡1, 3(mod 8), so there is a primary p which divides y^2+8, and p≡5 (mod 8), or p≡7 (mod 8). So $l = \frac{y^2}{p} = \frac{-8}{p} = \frac{-2}{p}$, is a contradiction.

Theorem. The equation $x^4+y^4=z^2$ (1) does not have solution in Z* [4].

Demonstration. Presuming that there is a solution in Z* of the equation (1). We can presume that this solution is of zero natural numbers. Like any non empty quantity of natural numbers has a small element, then among the solutions of equation (1) there is (x, y, z) with z minimum. When x or y have to be par, let us presume that x is even. Like $(x^2)^2+(y^2)^2=z^2$ and x^2,y^2 and z are natural (they can be relatively primary), and the natural numbers m, n with n<m, primary and of different parities so that $x^2=2mn$, $y^2=m^2-n^2$ and $z=m^2+n^2$. If m=2k and n=2t+1 then $y^2=4(k^2-t^2-t-1)+3$, which is not possible (because y^2 has to be 4k or 4k+1). So m is uneven and n is even. We have n=2q; and $x^2=4mn$ so that $mq = \left(\frac{x}{2}\right)^2$. When (m, n)=1 we deduce that $m = z_1^2$, $q=t^2$ with z_1 and t natural and primary among them .

Particularly, we observe that $y^2 = (z_1^2)^2 - (2t^2)^2 \Leftrightarrow y^2 + (2t^2)^2 = (z_1^2)^2$. There is a, b natural zero primary of different parities with $a > b$ so that $2t^2=2ab \leftrightarrow t^2=ab$, $y=a^2-b^2$, $z^2=a^2+b^2$. As a and b are primary and $t^2=ab$ we deduce that $a=x_1^2$, $b=y_1^2$ and $x_1^4+y_1^4 = z_1^2$. We deduce that (x_1, y_1, z_1) is the solution of equation (1) and depending of the choice of z we have $z_1 \geq z \leftrightarrow z_1^2 \geq z \leftrightarrow m \geq m^2 + n^2$ which is absurd.

Corollary. The equation $x^4+y^4=z^4$ (2) can not have solutions of full zero numbers [9].

Observations: 1. Equation(2) is a special case of Fermat $x^n+y^n=z^n$ (3). Where n>2 is a natural number and x,y,z are full zero numbers. Fermat theorem (the conjecture of Fermat) says that the equation (3) does not have solutions if n>2.

Theorem. The equation $x^4-y^4=z^2$ (4) does not have solutions in Z^* [9].

Demonstration. We presume that x, y, z >0 and (x, y)=1. We write the equation like $(x^2-y^2)(x^2+y^2)=z^2$. We easily see that $(x^2-y^2,x^2+y^2)=1$ or $(x^2-y^2,x^2+y^2)=2$. In the first case we get the system $\begin{cases} x^2+y^2=u^2 \\ x^2-y^2=v^2 \end{cases}$ which can not be solved in full zero numbers. In the second case we get $\begin{cases} x^2+y^2=2s^2 \\ x^2-y^2=8r^2 \end{cases}$ meaning $\begin{cases} s^2+(2r)^2=x^2 \\ s^2-(2r)^2=y^2 \end{cases}$ a system which, is again without solutions.

Example. Solve in full numbers the equation $x^4+y^4=2z^2$.

Solution. Without restraining the general we can presume that (x,y)=1. Then x and y are both uneven and

$z^4-(xy)^4=\left(\frac{x^4-y^4}{2}\right)^2$. We have xyz=0 or $x^4-y^4=0$, so x=y=z=0 or $x^2=y^2=z$. the solutions of the equation are (k,k,k^2), $k \in Z$.

2.3.2. Pell Equations

Thue's theorem If $f=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1 z+a_0$ a polynomial with full values degree ≥3, irreducible above Z. The homogeneous polynomial $F(x, y)=a_nx^n+a_{n-1}x^{n-1}y+\ldots+a_1xy^{n-1}+a_0y^n$. If m is a full non-null number, the equation F(x,y)=m does not have solutions or has a finite number of solutions in the multitude of full numbers.

Observation. This result contrasts with the situation when the level of F is n=2. In this case the full, if $F(x,y)=x^2-Dy^2$, where D is a natural number which does not have a perfect square, any zero number m, the general Pell equation $x^2-Dy^2=m$ either has no solutions, or has an infinity of solutions.

Short history

The first study on the equation $x^2-dy^2=1$, were done by Euler, the English mathematician John Pell. These should be called after Fermat, as he was the first to study the properties of un trivial solutions. The Pell equation date back from the Greeks. Theon of Smyrna approximated $\sqrt{2}$ through fractions x/y, where x and y are solutions of the equation $x^2-2y^2=1$.Generally, if $x^2=dy^2+1$, then $\frac{x^2}{y^2}=d+\frac{1}{y^2}$ and for y big enough, $\frac{x}{y}$ he approximates $\sqrt{d}$, known by Archimedes.

Archimedes's problem of bulls - finding the smallest group of bulls satisfying the seven conditions in eight unknown terms, was solved after more than 2000 years by Carl Amthov, in 1880. It reduces to finding a minimal solution of the equation $x^2-4729494y^2=1$, a solution where y has 41 terms [4].

The solving of the equation $x^2-dy^2=1$ was done by Diophantus in "Arithmetica". In the case when $d=m^2+1$, Diophantus found a particular solution $x=2m^2+1$, y=2m .

The Pell equations can be found in Hindu mathematics. In the 4th century, the Indian mathematician Baudhayana observed that x=577 , y=408 is a solution of the equation $x^2-2y^2=1$ and used fraction $\frac{577}{408}$ to approximate $\sqrt{2}$. In the 7th century, Brahmagupta determined the minimal solution of the equation $x^2-92y^2=1$. This is x=1151 , y=120. In the 12th century, the Indian mathematician Bhaskara showed that the minimal solution of the equation $x^2-61y^2=1$ is x=226153980 y=1766319049.

The study of Pell's equations was done by Fermat, and Wallis in 1657, Euler in 1770, Lagrange in 1766 etc.

2.3.3. Solving Pell Equations by Elementary Methods

Solving Pell Equation according to Lagrange

Theorem. If D is a natural number which is not a perfect square, the equation $u^2-Dv^2=1$ (1) has an infinity of solutions in natural numbers and the general solution is given by $(u_n,v_n)_{n\geq 1}$, where $u_{n+1}=u_0u_n+Dv_0u_n$, $v_{n+1}=v_0u_n+u_0v_n$, $u_1=u_0$, $v_1=v_0$ (2) (u_0 , v_0) is a fundamental solution, the smallest solution different from (1, 0) [10].

Demonstration Let us show that the equation (1) has a fundamental solution. Let there be c_1 a natural number bigger than 1. Let us show that there are natural numbers $t_1,w_1 \geq 1$ so that $\left|t_1-w_1\sqrt{D}<\frac{1}{c_1}\right|$, $w_1 \leq c_1$. By considering $l_k=[k\sqrt{D}+1]$, $k\in[0,c_1]\cap N$ we have $0\leq l_k-k\sqrt{D}\leq 1$ and $\sqrt{D}$ is an irrational number, and $l_{k'}\neq l_{k''}$ for k' ≠ k". There is i, j, p∈ $\{0,1,2,\ldots,c_1\}$, i ≠ j, p ≠ 0, so that $\frac{p-1}{c_1}<l_i-i\sqrt{D}\leq\frac{p}{c_1}$ and $\frac{p-1}{c_1}<l_j-j\sqrt{D}\leq\frac{p}{c_1}$ as there are c_1 intervals like $(\frac{p-1}{c_1},\frac{p}{c_1})$, p=1,$c_1$ and c_1+1 numbers like $l_k-k\sqrt{D}$, , $k\in[0,c_1]\cap N$.

From the inequalities above we have $\left|(l_i-l_j)-(j-i)\sqrt{D}\right|<\frac{1}{c_1}$ and writing $|l_i-l_j|=t_1$ and $|j-i|=w_1$ we deduce that $\left|t_1-w_1\sqrt{D}\right|<\frac{1}{c_1}$ and $w_1 < c_1$. By multiplying with $t_1-w_1\sqrt{D}\leq 2w_1\sqrt{D}+1$ we get $|t_1^2-Dw_1^2|<2\frac{w_1}{c_1}\sqrt{D}+\frac{1}{c_1}<2\sqrt{D}+1$.Choosing a natural number $c_2 > c_1$ so that $\left|t_1-w_1\sqrt{D}\right|>\frac{1}{c_1}$, we get natural numbers t_2 ,w_2 with properties like $|t_1^2-Dw_1^2|<2\sqrt{D}+1$ and $|t_1-t_2|+|w_1-w_2|\neq 0$.

By continuing this procedure, we get distinct pairs $(t_n,w_n)_{n\geq 1}$ which accomplish the inequalities $|t_1^2-Dw_1^2|<2\sqrt{D}+1$, for any natural number n. So the interval $(2\sqrt{D}-1 \; ,2\sqrt{D}+1)$ contains a k full zero number with the property that there is another line of $(t_n,w_n)_{n\geq 1}$ which accomplishes the equation $t^2-Dw^2=k$. This line has at least two pairs (t_s ,w_s), (t_r ,w_r) for which $t_s \equiv t_r(\text{mod } |k|)$, $w_s \equiv w_r(\text{mod } |k|)$, and $t_sw_r - t_rw_s \neq 0$ contrary we get $t_s=t_r$ and $w_s=w_r$, in contradiction with $|t_s-t_r|+|w_s-w_r|\neq 0$. Let there be $t_0=t_st_r-Dw_sw_r$ and $w_0=t_sw_r-t_rw_s$. Then $t_0^2-Dw_0^2=k^2$ (3). On the other side, $\mathrm{t_0=t_st_r-Dw_sw_r\equiv t_s^2-Dw_0^2\equiv 0(mod|k|)}$, implies

$w_0 \equiv 0(\mathrm{mod}|k|)$. The pair (t , w), where $t_0 = t|k|$ and $w_0 = w|k|$ is a nontrivial solution of Pell's equation (1).

Let us show that the pair (u_n, v_n) defined in (2) accomplishes Pell equation (1) for any n>0. If (u_n, v_n) is a solution of this equation, then $u_{n+1}^2 - Dv_{n+1}^2 = (u_0u_n + Dv_n)^2 - D(v_0u_n + Dv_n)^2 = (u_0^2 - Dv_0^2)(u_n^2 - Dv_n^2) = 1$ and (u_{n+1} , v_{n+1}) is a solution of the equation (1).

We observe that for any $n \geq 1$ there is the relation $u_{n-1} + v_{n-1}\sqrt{D} = \left(u_0 + v_0\sqrt{D}\right)^2$ Having relation $z_n = u_{n-1} + v_{n-1}\sqrt{D} = \left(u_0 + v_0\sqrt{D}\right)^2$ and observing that $z_1 < z_2 < \ldots < z_n <$ …… We will show that all the solution of equation (1) are like (4). If the equation (1) would have a solution (u , v) so that $z = u + v\sqrt{D}$ is not like (4), there is a natural number m with the property $z_m < z < z_{m+1}$. So $1 < \left(u + v\sqrt{D}\right)\left(u_m - v_m\sqrt{D}\right) < (u_0 + v_0\sqrt{D})$, and we get $1 < (uu_m - Dvv_m) + (u_mv - uv_m)\sqrt{D} < \left(u_0 + v_0\sqrt{D}\right)$.

On the other side we have $(uu_m\text{-}Dvv_m)^2\text{-}D(u_mv\text{-}uv_m)^2 = (u^2\text{-}Dv^2)\,(u_m^2 - Dv_m^2) = 1$, so (uu_m-Dvv_m, u_mv-uv_m) is a solution for (1) smaller than (u_0,v_0) , which contradicts the supposition of minimality of (u_0,v_0).

Observations. 1) Recurrent relations (3.2.1) can be written matrix like

$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} u_0 & Dv_0 \\ v_0 & Du_0 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$. Where we get

$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_0 & Dv_0 \\ v_0 & Du_0 \end{pmatrix}^n \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ (5). If $\begin{pmatrix} u_0 & Dv_0 \\ v_0 & Du_0 \end{pmatrix}^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Is known that a_n, b_n, c_n, d_n are linear combinations of λ_1^n, λ_2^n where λ_1, λ_2 are values of the matrix $\begin{pmatrix} u_0 & Dv_0 \\ v_0 & Du_0 \end{pmatrix}$ Taking into consideration (5) after a simple analysis we get

$$u_n = \frac{1}{2}[\left(u_0 + v_0\sqrt{D}\right)^n + \left(u_0 + v_0\sqrt{D}\right)^n ,$$

$$v_n = \frac{1}{2\sqrt{D}}[\left(u_0 + v_0\sqrt{D}\right)^n + \left(u_0 - v_0\sqrt{D}\right)^n \qquad (6).$$

2) the solutions of the Pell equation, given in one of the examples in (4) or(6), can be used in approximating the radicals of the natural numbers which are not perfect squares. If $\left(u_n, v_n\right)_{n\geq 1}$ are the solutions of equation (1), then $u_n - v_n\sqrt{D} = \frac{1}{u_n + v_n\sqrt{D}}$ so $\frac{u_n}{v_n} - \sqrt{D} = \frac{1}{v_n(u_n + v_n\sqrt{D})} <$ $< \frac{1}{\sqrt{D}v_n^2} < \frac{1}{v_n^2}$. We have $\lim_{n\to\infty} \frac{u_n}{v_n} = \sqrt{D}$. The fractions $\frac{u_n}{v_n}$ approximate $\sqrt{D}$ with an error smaller than $\frac{1}{v_n^2}$.

3. Elementary Methods of Solving

3.1. The Method of Decomposition

This method consists in writing the equation $f(x_1, x_2, \ldots, x_n)=0$ like $f_1(x_1,x_2, \ldots,x_n)$, $f_2(x_1,x_2,\ldots,x_n)$,…. $f_k(x_1,x_2,\ldots,x_n) = a$, where f_1 , f_2, …..,$f_k \in Z[X_1 ,X_2,\ldots X_k]$ and $a \in Z$. Using the decomposing method in primary terms of *a*, we get a finite number of decompositions in k full factors $a_1,a_2, \ldots,a_k$. Each decomposition of this type leads to a system of equations like :

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = a_1 \\ f_2(x_1, x_2, \ldots, x_n) = a_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_k(x_1, x_2, \ldots, x_n) = a_k \end{cases}.$$

By solving the system of equations we have the multitude of solutions for the given equation.

Example 1. Solve in full numbers the equation x + y = xy [10] .

Solution the equation is written like: (*x* - 1)(*y* - 1) = 1. As the product of two numbers is 1 if only both numbers are 1 or -1, we get the systems:

$\begin{cases} x - 1 = 1 \\ y - 1 = 1 \end{cases}$ and $\begin{cases} x - 1 = -1 \\ y - 1 = -1 \end{cases}$. With solutions like (0,0) and (2,2).

Example 2 Find all the solutions for the equation $(x^2+1)(y^2+1)+2(x-y)(1- xy)=4(1+xy)$.

Solution We write the equation like $x^2y^2 - 2xy+1+x^2+y^2 - 2xy + 2(x-y)(1-xy) = 4$ or $(xy-1)^2+(x-y)^2 - 2(x-y)(xy-1) = 4$. This is equivalent with $[xy-1-(x-y)]^2=4$ where we get $(x+1)(y-1)=\pm 2$.

If (x+1)(y-1)=2 we have the following systems of equations:

$$\begin{cases} x + 1 = 2 \\ y - 1 = 1 \end{cases}; \begin{cases} x + 1 = -2 \\ y - 1 = -1 \end{cases}; \begin{cases} x + 1 = 1 \\ y - 1 = 2 \end{cases}; \begin{cases} x + 1 = -1 \\ y - 1 = -2 \end{cases}$$

which lead to the solutions (1,2), (-3, 0) , (0, 3), (-2,-1).

If (x+1)(y-1)=-2, we get the systems:

$$\begin{cases} x + 1 = 2 \\ y - 1 = -1 \end{cases}; \begin{cases} x + 1 = -2 \\ y - 1 = 1 \end{cases}; \begin{cases} x + 1 = -1 \\ y - 1 = 2 \end{cases}; \begin{cases} x + 1 = 1 \\ y - 1 = -2 \end{cases}$$

whose solutions are (1,0), (-3, 2), (0,-1), (-2,3).

All the eight pairs are solutions for the given equation.

3.2. The Parametrical Method of Solving

In many situations the solutions of a Diophantine equation $f(x_1,x_2,\ldots,x_n)=0$ can be represented parametrically like $x_1 = g_1(k_1, k_2, \ldots, k_l)$, $x_2 = g_2(k_1, k_2, \ldots, k_l)$, ..., $x_n = g_n(k_1, k_2, \ldots, k_l)$, where $g_1,g_2,\ldots,g_n$ are functions of l-variables, with full values and $k_1,\ldots,k_l \in Z$.

For some Diophantine equations the multitude of solutions can have many parametrical representations.

In many cases, it is not possible to find all the solutions of a Diophantine equation. The parametrical method is a means to emphasize the infinite families of solutions.

Example. Show that there is an infinity of triplets (x, y, z) of full numbers so that $x^3+y^3+z^3= x^2+y^2+z^2$ [8].

Solution. By choosing l z=-y, the equation becomes $x^3=x^2+2y^2$. If y= mx, $m\in Z$, then $x=1+2m^2$ and we get the infinite family of solutions $x=1+2m^2$, $y=m(1+2m^2)$, $z=-m(1+2m^2)$, $m\in Z$.

3.3. Solving Diophantine Equations with Inequalities

This method consists in determining the intervals with unknown terms, by using the adequate inequalities. In general, this method leads not only to a finite number of possibilities for all the unknown terms or for part of these.

Example Determine all the pairs (x , y) of full numbers so that $x^3+y^3=(x+y)^2$ [2].

Solution First, we observe that the pairs (k, - k), k∈ Z are solutions for the given equation.

If $x+y \neq 0$, the equation becomes $x^2- xy+y^2=x+y$, which is equivalent with $(x-y)^2+(x-1)^2+(y-1)^2=2$. From this equality of full numbers we deduce $(x-1)^2 \leq 1$ and

$(y-1)^2 \leq 1$. Where $x\in [0,2]$, $y\in [0,2]$. We get the solutions (0 ,1) , (1, 0) ,(1 ,2) , (2 ,1) ,(2 , 2).

3.4. The Method of Mathematical Induction

The mathematical induction is a very much used and elegant method in demonstrating some affirmations which depend on the quantity of natural numbers.

Let there be $P(n), n \geq 0$ a series of propositions. This method helps us demonstrate that the proposition *P*(n) is true for any $n \geq n_0$, where n_o is a natural number.

Mathematical induction (weak form): We presume that

- P (n_o) is true;
- For any $k \geq n_o$, where P (k) is true we have P (k +1) which is true. Then, $P_{(n)}$ is true for any $n \geq n_0$.

The mathematical induction (with s step): Being s a natural number. We presume that

- P (n_o) , P (n_{o+1}),…, P (n_{o+s-1}) are true;
- For any $k \geq n_0$, where P (k) is true we have P (k+s) which is true. Then P(n) is true for any $n \geq n_0$.

Mathematical induction (strong form): We presume that P(n_0) is true; For any $k \geq n_0$, where P(m) is true for any m with $n_0 \leq m \leq k$, where P(k+1) is true. Then P(n) is true for any $n \geq n_0$.

This demonstration method is frequently used in different mathematical disciplines, including the theory of numbers. The following examples show the use of the mathematical induction in the study of Diophantine equations.

Example 1 Demonstrate that for each natural number $n \geq 3$, there are the uneven natural numbers x, y, with the property $7x^2+y^2=2^n$ [8].

Solution We will demonstrate that there are the natural uneven numbers x_n , y_n so that $7x_n^2 +y_n^2 =2^n$, $n \geq 3$. For n = 3 , we have $x_3 = y_3 = 1$.

Presuming that for an $n \geq 3$ fixed, there is x_n , y_n uneven which accomplish $7x_n^2 +y_n^2 =2^n$. We will build up the pair (x_{n+1} ,y_{n+1}) of natural uneven numbers so that $7x_{n+1}^2 +y_{n+1}^2 = 2^{n+1}$. We observe that

$$7\left(\frac{x_n \pm y_n}{2}\right)^2 + \left(\frac{7x_n \mp y_n}{2}\right)^2 = 2(7x_n^2 + y_n^2) = 2^{n+1} .$$

One of the natural numbers $\frac{x_n+y_n}{2}$ and $\frac{x_n-y_n}{2}$ is uneven (because the sum of the two numbers is max $\{x_n, y_n\}$, which is uneven). If for example $\frac{x_x+y_n}{2}$ is uneven, then $\frac{7x_n-y_n}{2} = 3x_n + \frac{x_x-y_n}{2}$ is also uneven (like the sum of an uneven and an even number), we could choose $x_{n+1}= \frac{x_n+y_n}{2}$ and $y_{n+1}= \frac{7x_n-y_n}{2}$.

If $\frac{x_n-y_n}{2}$ is uneven, then $\frac{7x_n+y_n}{2} = 3x_n + \frac{x_x+y_n}{2}$ and we can choose $x_{n+1}= \frac{|x_n-y_n|}{2}$ and $y_{n+1}= \frac{|7x_n-y_n|}{2}$.

3.5. The Method of Modular Arithmetic

In many situations, the simple examples of modular arithmetic prove to be useful in demonstrating that some Diophantine equations do not have a solution or the possibility of solutions is reduced.

Example 1 Show that the equation $(x+1)^2+(x+2)^2 +\ldots+ (x+2001)^2 = y^2$ does not have a solution.

Solution Let there be x=z-1001. The equation becomes $(z-1000)^2 +\ldots+ (z-1)^2 +z^2 + (z+1)^2 + \ldots+ (z+1000)^2 = y^2$ or $2001 z^2 +2(1^2 +2^2 +\ldots 1000^2)=y^2$. It follows $2001z^2 + 2\cdot[(1000\cdot1001\cdot2001):6]=y^2$. Or the equivalent $2001 z^2 + 1000\cdot1001\cdot667 = y^2$. The left side of the last relation is congruent with 2 (mod 3), which can not be a perfect square.

Example 2 Determine all the pairs of primary numbers (p,q) so that $p^3 -q^5 =(p+q)^2$ [2].

Solution The only solution is (7, 3). Presuming, for the beginning, that none of the primary numbers *p* and *q* is equal to 3. Then $p\equiv 1$ or 2 (mod 3) and $q \equiv 1$ or 2 (mod 3). If we have $p\equiv q$ (mod 3), then the left side of the equation can be divided by 3, and the right side would not have this property. The same happens if $p \neq q$ (mod 3).

If p = 3,then $q^5 < 27$, which is not possible.

If q = 3, we get $p^3 - 243 = (p+3)^2$, an equation with unique full solution p =7.

3.6. Fermat's Method of Infinite Descent

Fermat's method of infinite descent (FMID) can be formulated as follows: Let there be k a natural number. Presuming that: If P(m) is true for a number m > k , then there is a smaller number j ,m > j > k with the condition that the proposition P(j) is true. Then P(n) is false for any n > k. This happens when n > k for which the proposition P(n) is true, then we could form an infinite series of natural numbers $n > n_1 > n_2 > \ldots$, which is not possible. Two particular cases of FMID are very useful in the study of Diophantine equations [9].

FMID- Variant 1: There is no strict decreasing series of natural numbers $n_1 > n_2 > \ldots$ In some situations, it is convenient to replace FMID- Variant 1 with the equivalents: If n_0 is the smallest natural number n for the proposition P(n) is true, then P(n) is false for any $n < n_0$.

FMID - Variant 2: If the line of natural numbers $(n_i)_{i\geq1}$ accomplishes the inequalities $n_1 \geq n_2 \geq \cdots$, then there is i_0 so that $n_{i_0} = n_{i_0+1} = \cdots$

Example 1 Solve in natural numbers the equation $x^3+2y^3=4z^3$ [5].

Solution Observe that (0,0,0) is a solution. We will show that there are no other solutions. Presuming that the equation has a nontrivial solution (x_1,y_1,z_1). As both numbers $\sqrt[3]{2}, \sqrt[3]{4}$ are irrational, it is not difficult to see that $x_1 > 1, y_1 > 1, z > 1$.. From the relation $x_1^3 + 2y_1^3 = 4z_1^3$ it results $2|x_1$, so $x_1=2x_2$, $x_2 \in Z_+$. Then $4x_2^3 + y_1^3 = 2z_1^3$, so $y_1=2y_2$, $y_2 \in Z_+$. Analogically, we get $z_1=2z_2$, $z_2 \in Z_+$. Building a "new" solution (x_2, y_2, z_2) with the property $x_1>x_2$, $y_1>y_2$, $z_1>z_2$. Continuing this procedure, we get a line of natural solutions $(x_n, y_n, z_n)_{n\geq 1}$ with the property that $x_1>x_2>x_3>\ldots$ which contradicts FMID – Variant 1.

Example 2 Solve in natural numbers the equation $2^x - 1 = xy$ [5].

Solution We remark that (0, k), $k \in \mathbf{Z}_+$ and (1, 1) are solutions to the equation. Applying FMID to the primary factors of x, we will show there are no natural solutions.

Being p_1 a divisor prim of x and q the smallest natural number with the property $p_1|2^q - 1$. According to Fermat's theory, there results $p_1|2^{p_1-1}$, and so $q \leq p_1 - 1 < p_1$.

Let us show that q|x. If not true, we get x= kq+r where 0<r<q. So $2^x - 1 = 2^{kq}2^r - 1 = (2^q)^k \cdot 2^r - 1 = (2^q - 1+1)^k \cdot 2^r - 1 \equiv 2^r - 1 \pmod{p_1}$.

It results that $p_1/2^r-1$, which contradicts the minimality of q. So q/x and $1<q<p_1$. Considering now p_2 a primary divisor of q. Evidently p_2 is a divisor of x and $p_2 < p_1$. By continuing this procedure, we get an infinite descending series of the primary divisors of x.

4. Illustrated Application

1) E: 12000, Determine x, y∈N so that $x^3-y^3=5y^2+58$

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Solution: As $5y^2+58>0$, it results $x^3 > y^3$ and so x>y. Being n∈N*, x=y+n. The equation becomes $(y+n)^3 - y^3 = 5y^2+58 \Leftrightarrow y^2(3n-5)+3yn^2+n^3=58$.

For n=1, the equation becomes $-2y^2+3y-57=0$, which has no real solutions. For n=2, the equation $y^2+12y-50=0$ has the solutions $-6 \pm \sqrt{86} \notin N$. For n=3, the equation $4y^2+27y-31=0$ has the solutions y=1, and $y = -\frac{31}{4} \notin N$.

For n ≥ 4 , the equality is not satisfying, whichever is y∈N.

In conclusion, y=1 and x=4 is the only solution.

2) Solve in natural numbers the equation : x+y+z=xyz+3

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Solution: At least one of the natural numbers x, y or z which verifies the considered equation has to be 0. If x, y, z∈N* then (x-1)(y-1)≥0, meaning: xy≥x+y-1 (1).

Analogically, we get xz≥ x+z-1 (2). And yz ≥y+z-1 (3).

By multiplying the relation (1) with z we get: xyz≥xz+yz-z, where, using (2) and (3), we deduce: xyz≥x+z-1+y+z-1-z, meaning: xyz≥x+y+z-2, where: xyz+3≥x+y+z, which contradicts the equality xyz+3=x+y+z.

So, at least one of the natural numbers x,y,z has to be 0. If x=0, we get y+z=3, where the solutions are: (0,0,3) (0,1,2) (0,2,1) and (0,3,0), analogically for y=0, respectively z=0, we get the solutions: (0,0,3), (0,1,2), (0,2,1), (0,3,0),(1,0,2) ,(2,0,1) ,(3,0,0) ,(1,2,0) and (2,1,0).

3) Let us determine the multitude of the triplets (a,b,c) of natural zero numbers whose roots of equations $ax^2-bx+c=0$ and $ax^2-cx+b=0$ are natural numbers.

M. Chiriță Mathematics Olympics, Local Stage, IX[th] grade, Bucharest, February, 2001.

Solution: Being x_1, x_2∈N* the roots of the first equation and x_3, x_4∈N* the roots of the second equation. According to Viette relations, $\frac{b}{a} = x_1 + x_2 \in N$ and $\frac{c}{a} = x_1 \cdot x_2 \in N$, meaning that a divides b and a divides c. Writing $p = \frac{b}{a}$ and $q = \frac{c}{a}$, p, q ∈N*, the equations become: $x^2-px+q=0$ and $x^2-qx+p=0$.

As x_1, x_2 and x_3, x_4∈N*, we have $(x_1-1)(x_2-1) \geq 0 \Rightarrow x_1x_2+1 \geq x_1+x_2 \Rightarrow p+1 \geq q$ and $x_3x_4+1 \geq x_3+x_4 \Rightarrow q+1 \geq p$. So p+1 ≥ q ≥ p-1 or q∈{p+1, p, p-1}. We distinguish the cases:

a) q=p. The given equations come back to $x^2-px+p=0$. Then the discriminant $\Delta=p^2-4p$ is a perfect square. Writing $\Delta=k^2$, k∈N, we have $k^2=p^2-4p \Leftrightarrow 4=(p-2)^2-k^2 \Leftrightarrow 4=(p-2-k)(p-2+k)$. As the numbers p-2-k∈Z and p-2+k∈Z have the same parity, it results that p-2-k= p-2+k, so p=4. We have the equation $x^2-4x+4=0$ with natural solutions. In conclusion, p=q=4.

b) q=p+1. The equations become: $x^2-px+p+1=0$ and $x^2-(p+1)x+p=0$. The second equation has natural solutions 1 and p. For the first equation we have $\Delta= p^2-4p-4=(p-2)^2-8$. Writing $\Delta=k^2$, k∈N, we get 8=(p-2-k) (p-2+k) and if p-2-k< p-2+k are natural numbers of the same parity, we have p-2-k=2 and p-2+k=4 where p=5. The equation $x^2-5x+6=0$ has natural solutions 2 and 3. In conclusion, p=5 and q=6.

c) q=p-1⇒p=q+1 and like in b) we get q=5 ad p=6. The triplets (1, p,q) are: (1,4,4),(1,5,6) and (1,6,5), so the triplets (a,b,c) =(a,ap,aq) are: (a,4a,4a) ,(a,5a,6a) and (a,6a,5a), a∈N* Solve the equation in natural numbers: x+y+z=xyz+3.

4) Solve in Z the equation: $\left||x+1|+2\right| = 4$

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*Solution:*From $|x| = a\ (a \geq 0) \leftrightarrow x = \pm a$ results $\left||3x+1|+2\right| = 4 \Leftrightarrow |3x+1|+2 = 4$ or
$|3x+1|+2 = -4 \Leftrightarrow |3x+1| =$ or $|3x+1| = -6$ (impossible). $|3x+1| = 2 \Leftrightarrow 3x+1=\pm 2 \Leftrightarrow 3x+1= 2$ or 3x+1= -2 ⇔ 3x=1 or 3x=-3. As x∈Z, it results that x= -1 is the only solution of the equation.

5) E: 12041. Determine a, b∈Z so that $\frac{1}{a} + \frac{1}{b} - \frac{1}{ab} = \frac{1}{2}$.

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Solution: The equation is written like 2(b+a-1) =ab ⇔ -2 = ab-2a-2b⇔2=ab-2a-2b+4 ⇔ 2 = (a-2)(b-2).We have the following cases :

$$a)\begin{cases} a-2=1 \\ b-2=2 \end{cases} \Rightarrow \begin{cases} a=4 \\ b=3 \end{cases};\ c)\begin{cases} a-2=-1 \\ b-2=-2 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=0 \end{cases},$$

it is not convenient and,

$$b)\begin{cases} a-2=2 \\ b-2=1 \end{cases} \Rightarrow \begin{cases} a=3 \\ b=4 \end{cases};\ d)\begin{cases} a-2=-2 \\ b-2=-1 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=4 \end{cases}$$

it is not convenient.

6) E: 12022. Determine x, y∈Z with the property that $x^3+y^3=x^2+y^2$.

Solution: Note that $x^2+y^2\geq 0 \Rightarrow x^3+y^3 \geq 0$, so it is not possible that x and y to be simultaneous negative. There are the following cases.

a) x≥0, y≥0. Then $x^3\geq x^2$ (the equality being possible only for x=0 or x=1) and $y^3\geq y^2$(the equality is for y=0 or y=1)⇒ $x^3+y^3\geq x^2+y^2$, with equality only for $(x,y) \in \{(0,0); (0,1); (1,0); (1,1)\}$.

b) x≥0>y. We write z=-y so z >0. The equation becomes $x^3-z^3= x^2+z^2$, as $x^2+z^2\geq 0$, results x ≥ z. Being n∈N, n=x-z. Then, the equation is written $n(x^2+xz+z^2)= x^2+z^2$. If n=0,then $x^2+z^2=0\Rightarrow x=z=0$, false, as z >0

If n≥1, then $n(x^2+xz+z^2) \geq x^2+xz+z^2>x^2+z^2$ (because xz>0 from x-z=n≥1 and z>0). In conclusion, there are no solutions in this case. Analogically, we study the case of y≥0>x.

7) E:11594. Let us demonstrate that there is no x,y,z∈Z so that there is the equality: $x^4+y^4+z^4=x^3+y^3+z^3+5$.

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Solution: The equation is written: $(x^4-x^3)+(y^4-y^3)+(z^4-z^3)=5$ or $(x-1)x\,x^2+(y-1)yy^2+(z-1)zz^2=5$.

Because x-1 and x are full consecutive numbers, the result (x-1)x is an even number and so x^4-x^3 is even. Analogically y^4-y^3 and z^4-z^3 are even, so the left term is an even number. When 5 is uneven, the equation has no full solutions.

8) Let there be p and q two primary numbers. Solve in natural numbers the equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{pq}$.

Solution : The equation is equivalent with the algebraic Diophantine equation $(x-pq)(y-pq)=p^2q^2$. Considering all the positive divisors of the number p^2q^2 we get the following systems of equations:

$$\begin{cases} x-pq = 1 \\ y-pq = p^2q^2 \end{cases};\ \begin{cases} x-pq = p \\ y-pq = pq^2 \end{cases};\ \begin{cases} x-pq = q \\ y-pq = p^2q \end{cases};$$

$$\begin{cases} x-pq = pq \\ y-pq = pq \end{cases};\ \begin{cases} x-pq = p^2 \\ y-pq = q^2 \end{cases};\ \begin{cases} x-pq = p^2q^2 \\ y-pq = 1 \end{cases},$$

which lead to the solutions: (1+pq,pq(1+pq)), (p(1+q),pq(1+q)), (q(1+p),pq(1+p)), (p(p+q),q(1+p)), (2pq,2pq), (pq(1+q),p(1+q)), (pq(1+p),q(1+p)), (q(p+q),p(p+q)) , (pq(1+pq),1+pq) .

Observation The equation $\frac{1}{x}+\frac{1}{y}=\frac{1}{n}$, where $n=p_1^{\alpha_1}p_2^{\alpha_2}\ldots\ldots p_k^{\alpha_k}$, has $(1+2\alpha_1)(1+2\alpha_2)\ldots.(1+2\alpha_k)$ solutions in full positive numbers. The equation is equivalent with $(x-n)(y-n)=n^2$ and $n^2=p_1^{2\alpha_1}p_2^{2\alpha_2}\ldots\ldots p_k^{2\alpha_k}$, has $(1+2\alpha_1)\ldots(1+2\alpha_k)$ positive divisors.

9) Find all the triplets (x, y, z) of natural numbers so that $x^3+y^3+z^3-3xyz=p$, where p is a primary number bigger than 3.

Solution: The equation is equivalent with $(x+y+z)\cdot(x^2+y^2+z^2-xy-yz-zx)=p$. Because x+y+z>1,

it results that x+y+z=p and $x^2+y^2+z^2-xy-yz-zx=1$. The last equation is equivalent with $(x-y)^2+(y-z)^2+(z-x)^2=2$. Without restraining the generality we can presume that x ≥ y ≥ z , we have x-y ≥ 1 , y-z ≥ 1 and x-z ≥ 2 , which implies $(x-y)^2+(y-z)^2+(z-x)^2 \geq 6>2$. So x=y=z+1 or x-1=y=z.

The primary number p is 3k+1 or 3k+2, in the first case the solutions are $(\frac{p-1}{3},\frac{p-1}{3},\frac{p+2}{3})$ and the corresponding movements. In the second case, the solutions are $(\frac{p-2}{3},\frac{p+1}{3},\frac{p+1}{3})$ and the corresponding movements.

10) Show that the equation $x^5 - y^2 = 4$ has no solutions in full numbers.

Solutions Consider the equation (model 11). Because $(x^5)^2= x^{10} \equiv 0$ or 1 (model11) for any $x^5 \equiv -1$, 0 or 1 (model 11). So $x^5 - 4 \equiv 6,7$ or 8 (model 11). On the other hand, $y^2 \equiv 0,1,2,3,4$ or 9 (model 11), so the equation as no full solutions.

11) Demonstrate that n is a natural zero number with the property that the equation $x^3 - 3xy^2 + y^3 = n$ has a solution (x, y) in full numbers, then it has at least three solutions in full numbers. Show that the equation has no solutions if n = 2891.

Solution: The left side of the equation looks like

$x^3 - 3xy^2 + y^3 = 2x^3 - 3x^2y - x^3 + 3x^2y - 3xy^2 + y^3 =$
$= 2x^3 - 3x^2y + (y - x)^3 = (y - x)^3 - 3(y - z)(-x)^2 + (-x)^3$.

This shows that if (x ,y) is a solution of the equation, then the same property applies to the pair (y-x,-x). Moreover, these two solutions are distinct because the relations y –x = x and –x =y lead to x =y = 0. Analogically: $x^3 - 3xy^2 + y^3 = x^3 - 3x^2y + 3xy^2 - y^3 + 2y^3 + 3x^2y - 6xy^2 = (x-y)^3 + 3xy(x-y) - 3xy^2 + 2y^3 = (-y)^3 - 3(-y)(x-y)^2 + (x-y)^3$. So (-y , x-y) is the third solution of the given equation.

We use the transformations (x, y) ⟶(y –x , - x), (x, y) ⟶(-y , x-y) to solve the second part of the problem. Presuming that the equation has solutions and we consider (x ,y) a solution of the equation. As 2891 is not divisible by 3 results $x^3 + y^3$ which is not divisible by 3. So, both numbers x, y give the same difference ≠ 0 in dividing by number 3, or one of them is divisible by 3 . Both situations imply that one of the numbers - x, y ,x –y is divisible by 3.

So $x^3 \equiv 2981$ (model 9), which is possible as any cube is congruent with 0 ,1 or 8 (model 9).

12) Determine all the primary numbers p for which the system of equations $p+1=2x^2$, $p^2+1=2y^2$. Has solutions in full numbers x, y.

Solution We will show that the only primary number with this property is p=7. Without restraining the generality we can presume that x, y_0. From $p^2+1=2x^2$ it results p=2. Moreover, $2x^2 \equiv 1 \equiv 2y^2$ (model p), implies $x \equiv +y$ (model p), as p is uneven. As c<y<p, we have x+y=p and so $P^2 +1= 2(p-x) = 2p^2 - 4px + p + 1$. Meaning that p=4x-1, $2x^2=4x$. It results x=0 or x=2, which leads to p= -1 or p=7. We get p=7 and the solution of the system is (x, y)=(2,5).

5. Research Findings

Therefore, the aim of our experiment was to investigate the efficiency of the classes of Diophantine equations in solving these equations in classes of excellence in secondary schools.

The obtained data proved that different situations created in the experiment represent valences specific to different categories of undergraduates (very good, good, mediocre).

Confronting students with different situations and solving problems using the logical deduction is a means of discovery, which mobilizes the pupils even more.

Thus, the hypothesis of my work was confirmed and I was able to reveal the existence of the possibility of creating characteristic situations specific to the teaching activities which take into consideration the amplification of inner reasons such as changing some extrinsically reasons into intrinsic reasons.

At the same time, the results of investigation confirm the hypothesis that if we use and turn to good account the in all lesson stages while teaching mathematics in the secondary school then all the lessons will be efficient and the best results of the pupils. The results obtained through the use of the proofs led to the following findings: - the demonstrations and their use belong to the motivational situation being efficient because they mobilize the undergraduates and the students when they teach; - the obtained data demonstrated that the results are superior in all tests with the various methods of solving Diophantine in activity of solving mathematical problems; - they activated the undergraduates with poor results too, eliminating their fear, shyness, discouragement; - any notion introduced or consolidated with their help is easier accessible contributing to the formation of abilities and skills for the demonstration of problems through the solving of Diophantine equations .

Using the logical deduction as an active participating method in solving Diophantine equations offers the possibility to the future teachers to know pupils better, to know their individual particularities, their proper style, intelligence, will, temperament, behavior, briefly their personality.

I consider that the proposed objective and the hypothesis of my research were confirmed and the importance of the teaching-learning elements solving of Diophantine equations through the deductive method is one of the most efficient methods of demonstrating and solving theoretical and practical problems. My strategy of presenting the way the undergraduates took part into solving the tasks is an attempt to use the theoretical and practical knowledge from reference works of combined with my experience.

5.1. Participants

The experiment took place during the second semester of the 2012-2013 school year, involving 52 students in the VIIth grade middle school.

The experimental group of 26 children (15 girls and 11 boys) from the National Pedagogical College Bacau, believed to be a representative sample of realization our research objectives.

The control group consisted of 20 students of the Faculty of Mathematics (10 girls and 10 boys) from the same college. In both classes, students are talkative, sociable, and their intellectual level is similar.

5.2. Research Methods and Techniques

The research was of an experimental type, using the test method. Other research methods and techniques used were: Pedagogical observation; The communication; Analysis of school documents and student work products; The interview; - Statistical techniques for data processing.

5.3. The Hypothesis

The research started from the following hypothesis: If we use the various methods of solving Diophantine in the activity of solving mathematical problems and demonstration of active and participatory teaching methods, all lessons will be effective and student achievement will contribute more to improving learning performance, increased efficiency and creativity of students.

5.4. Objectives of the Research

Students practitioners have established the following objectives: To show that regardless of the field, solving Diophantine equations using various methods should characterize man, in many cases, school, family and society; - Performing an empirical research on the use of experimental improvement of methods for solving Diophantine equations and combining traditional methods of teaching and learning with active-participative methods; Promoting the idea that using the methods of solving Diophantine equations students can develop thinking, creativity, feelings and positive attitude, competitive spirit, intellectual skills.

6. Presentation of the Final Test

PART I. Write the letter of the correct answer (45 points).

5p.1. Solve in integers the equation set $\frac{1}{2x}+\frac{1}{3y}=\frac{1}{4}$ [6]. Equation solutions are?:

A. (3,4); B. (3,2); C. (6,2); D. (-6,1).

5p.2. The least common multiple of numbers 12, 18 and 24 is: A. 6; B. 48; C. 72; D. 108.

5p.3. Natural solutions of the equation are xy-4x-3y+11=0 is: A. (2,3); B. (1,2); C.(4,5); D. (0,3).

5p. 4. If a group of history students obtain the following marks: 8, 7, 9, 10, the average mark will be: A. 7; B. 8; C. 9; D. 10.

5p.5. A person who deposits the sum of 1200 USD in a bank account with 10% interest per annum will have, after a year: A. 1210 USD; B. 2400 USD; C. 1320 USD; D. 2400 USD.

5p.6. As the ratio of the acute angles of an isosceles triangle is equal to: A. 1; B. 0.5; C. 0; D. 0, (3).

5p.7. The measured angles of a triangle are proportional to the numbers: 1, 2, 3. Triangle is: A. isosceles; B.

equilateral; C. angle greater than 90^0; D. rectangular.

5p.8. An isosceles triangle has 5cm and 12cm sides respectively. The perimeter is equal to: A. 22cm; B. 17cm; C. 29cm; D. 24cm.

5p.9. An isosceles triangle ABC is m (∡ ABC) = 1200 BC and AD ⊥, B ∈ (DC). AD = 6cm when AC is equal to: 36cm; B. 12cm; C. 18cm; D. 6cm [6].

PART II. The following problems require comprehensive solutions. (45 points)

9p.10. Solve in integers the equation set: xy+3x+2y+5=0.

9p.11. In the stadium there is a group of between 50 and 100 athletes that can be grouped in columns of 12,18 or 24, but every time this is attempted, there are six athletes who cannot form a complete column. What is the minimum number of athletes? [7].

9p.12. Solve the equation in the set of integers: $x^2+y^2+15=4(2x+3y)$ [6].

9p.13. A right angled triangle ABC has a hypotenuse BC = 12 cm and the side AC = 6cm. Let M be the midpoint of the hypotenuse and AD ⊥ BC, D ∈ (BC). Calculate the length of the segment DM.

9p.14. Let ABC be an isosceles triangle with base BC and M and N means that the sides AB, AC. Show that CM = BN [7].

The skills assessment test associated with the final assessment for Class VII a.

C1. Operations with integers and whole

C2. Equations Diophantine in the many N and Z.

C3. Using properties, proportions, percentages, use of the arithmetic mean, solve percentage problems.

C4. Sum of the angles of a triangle. Properties of isosceles triangle. Solve problems.

C5. Theorem sharing the rest. divisor common, multiple.

Table 1. *Scale of assessment.*

Part I

Item number	**I1**	**I2**	**I3**	**I4**	**I5**	**I6**	**I7**	**I8**	**I9**
Results	A, C,D	C	A, C	B	C	A	D	C	B
Score	5p	5p	5p	5p	5p	5p	5p	5p	5p

Part II

10	**(x+2)(y+3)=1 x=-1,y=2 or x=-3, y=-4**	**5p 4p**
11.	$n=12c_1+6; n=18c_2+6; n=24c_3+6$; n-6=72, n=78	5p 4p
12.	$(x-4)^2+(2y-3)^2=10$; (x,y) belongs to the crowd: {(5,3);(7,2);(3,3);(7,1);(5,0);(3,0); (1,2); (1,1)}	5p 4p
13.	AM=(BC/2)=6cm=AC=MC=BM, DM=3cm.	5p 4p
14.	Triangle ABC congruent to triangle BCD so BN congruent to CM	5p 4p

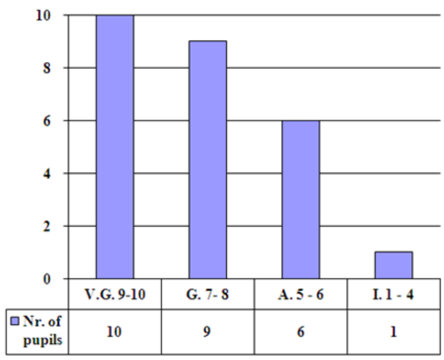
Table 2. *Matrix specifications - Test final assessment.*

CONTENTS/ SKILLS	**C1**	**C2**	**C3**	**C4**	**C5**	**TOTAL**
Operations with integers and whole	-	-	I5 (5p)	-	I4 (5p)	10p
Equations Diophantine in N and Z	I1 (5p)	I3 (5p)	I10 (9p)	I13 (9p)	-	28p
Sets	I2 (5p)	-	I6 (5p)	-	-	10p
Reports and proportions	-	-	-	-	I7 (5p)	5p
Triangle properties	-	-	-	I14 (9p)	I11 (9p)	18p
Congruence of triangles	-	I8 (5p)	I12 (9p)	I9 (5p)	-	19p
TOTAL	10p	10p	28p	23p	19p	90p

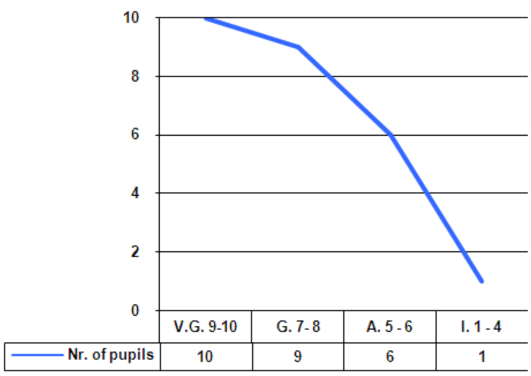
Table 3. *Test results that reflect student performance on the final test determined the overall mean 7.50.*

Note	**Ratings**	**Frequency**
9 - 10	very good	10
7 - 8	good	9
5 - 6	average	6
1 - 4	insufficient	1

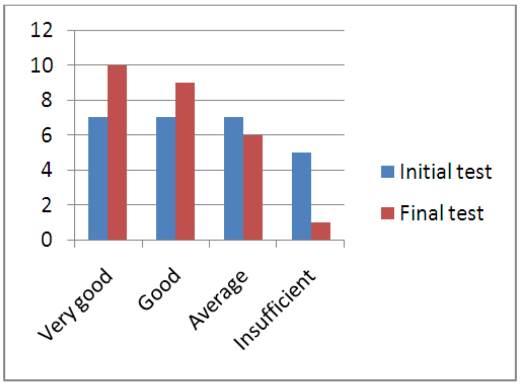
The results obtained through the use of the proofs led to the following findings.



Graphic 1. *Histogram reflecting the final test results of students.*



Graphic 2. *Frequency polygon reflecting the final test results of students.*



***Graphic 3.** Graph comparing the initial and final test.*

7. The Results and their Interpretation

Through the experiment carried out on an initial test with second year Math undergraduates of “Vasile Alecsandri” University of Bacau, it was proved that the teaching and the development of skills and abilities for assessment in high school are possible if we use various evaluation methods and procedures. This information was very useful in planning the following activities, taking into account the specificities of each student. Motivation for team learning consists (without the students being aware) of exciting activities, attractive, intuitive special materials, worksheets and modern teaching methods.

In terms of the second year students of the Faculty of Mathematics, it was found that through impact assessment, observed learning and assessment records, there was active participation on the part of the students, increasing the degree of intellectual effort, interest and curiosity with regard to mathematics.

This data was recorded in an observation grid. At the same time, the investigation results confirm the hypothesis that if we use various techniques for evaluation in all lesson stages, the teaching of mathematics in school will be more efficient, and the results of the pupils will improve.

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This data was recorded in an observation grid. At the same time, the investigation results confirm the hypothesis that if we use various techniques for evaluation in all lesson stages, the teaching of mathematics in school will be more efficient, and the results of the pupils will improve.

8. Conclusion

The article offers the conceptual frame and the necessary methodological references for the future secondary school and Maths teachers training through the initial formation.

The material is useful too, for those preparing their Master Degree and for those who are at the beginning of their career or those preparing to become in-service teachers, or those preparing their second degree exam.

The general problems of psycho pedagogy and the ones specific to the process of teaching, learning and assessment in Mathematics are coherently connected being based both on a vast documentation and on the author’s didactic experience.

The utility of the article is increased by the great number of adequate examples from the method of solving Diophantine equations and from the educational practice, examples which offer to the undergraduates help for their formation.

The article presents in a general way the deductive method of mathematical demonstration, after which some representative examples are offered.

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