A result in the theory of determinants from a semiotic viewpoint

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Abstract: We present a conceptual proof of the Cauchy-Binet theorem about determinants to show how much one can gain by investing a bit more in conceptual development, comparing this treatment with the usual one in terms of laborious matrix calculations. The purpose is to stimulate a conceptual understanding and to overcome the usual empiricism, which is an obstacle to a real understanding of mathematical knowledge. The article also aims to show that mathematical terms could be understood as dynamic processes, based on the interaction between intensional and extensional aspects. As it is not really possible to answer any question about the nature of mathematical objects definitively, much less to limit the possible interpretations of mathematical concepts, processes of concept evolution are of great importance to mathematics as a human activity.

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1. Complementarity

Let us go to the notion of complementarity, by pointing out once more that we use our symbols and concepts in a twofold sense, both attributively and referentially. Bertrand Russell illustrates the point by means of the distinction he draws between names and descriptions. We have, he writes:

. . . "two things to compare: a *name*, which is a simple symbol, directly designating an individual which is its meaning (or referent), and having this meaning in its own right independently of the meanings of all other words; a *description*, which consists of several words, whose meanings are already fixed, and from which results whatever is to be taken as the 'meaning' of the description. (Russell (1998), p. 174)

On account of this distinction between naming and describing Russell is led to criticize and refine Frege's interpretation of $A = B$ or of $A = A$. Frege treated the difference between these two forms of an equation by his own distinction between sense and meaning, concluding that singular descriptions function like designations, as one usually understands them referentially. Russell considers

this an error, for we cannot gain knowledge by just giving things new names.

Thus so long as names are used *as* names, 'Scott is Sir Walter" is the same trivial proposition as "Scott is Scott". (Russell (1998), p. 175)

And, *"*. . . a proposition containing a description is not identical with what that proposition becomes when a name is substituted, even if the name names the same object as the description describes. "Scott is the author of *Waverley*" is obviously a different proposition from "Scott is Scott". (Russell (1998), p. 174)

And by the very same token: "If 'x' is a name, x = x is not the same proposition as "the author of *Waverley* is the author of *Waverley*." . . . In fact, propositions of the form "the so-and-so is the so-and-so" are not always true: it is necessary that the so-and-so should *exist*. It is false that the present king of France is the present king of France, or that a round square is a round square" (Russell (1998), p. 176).

'Unicorn' would then be an abbreviated description and 'square root of -1' as well. For these descriptions the affirmation 'x exists' makes sense, although it may be false, whereas, according to Russell or Frege, 'y exists' is meaningless if 'y' is a name, because 'exists' is not a predicate.

But the essential point is that both, indices (names) as well as icons (predicates or descriptions) are essential although we may never be able to separate them completely, as we always use our linguistic terms both referentially and attributively.

To illustrate the latter point let us discuss the following example. Suppose an English tourist visiting Amazonia sees a biggish animal near the shore of a lake and asks what kind of animal this is. He is told that what is seen is a *Capivara*. As the tourist cannot speak Brazilian Portuguese this is only an indexical or referential designation, which leaves him without any representation for the moment. If he is offered, to relieve his frown, an Anglicization in the form of 'water hog', his face lights up and he says 'aha', actually believing to have understood what it is, the fact being that he is able to link something meaningful with the words of 'water' and 'hog'. This is thus a case of some kind of descriptive designation, which has the disadvantage, however, of creating completely false notions. For the *Capivara* is no swine at all, but a grass-eating rodent. The Amazonian is in the opposite situation, as for him the Indian name of *Capivara* has the meaning of 'grass-eater', while the designation 'water hog' tells him absolutely nothing.

Now such a referential use sometimes serves as the starting point of further observations if a motive or curiosity results. After some time, the tourist may observe some characteristics and habits of the *Capivara*, and then will be able to say, "*Capivaras* are good swimmers and divers", or "the *Capivara* lives in family groups", etc. Gradually, the use of the term changes and it is transformed into a description.

The key thing about a name or an index is that it has a direct connection with its object. In the case of the present example, this connection is established by concrete ostentation. It indicates its objects without giving any information about them. Therefore, we are able to understand an index as a *sign* only by means of some 'collateral experience', or contextual acquaintance with what the sign denotes to make the interpretation work. For instance, I point my finger to what I mean, but I cannot make my companion know what object I mean, if he cannot see it, or if seeing it, it does not, to his mind, separate itself from the surrounding objects in the field of vision.

The interdependence of attributive vs. referential uses of terms is much more prominent with respect to mathematical concepts than in empirical ones, because mathematical objects firstly do not exist independently of any representation and secondly because their instrumental character is much more pronounced.

If one for instance wants to know what "uneven number means the best thing is to answer by saying "x is an odd natural number if there exists some natural number n such that x = 2n+1".

Such an explanation serves well if for instance on tries to prove that the product of two uneven numbers is uneven again, because the proof becomes a straightforward calculation. On such grounds, Kant had called mathematics synthetic a priori. The proof does not come about by trying to interpret what the terms really mean.

Kant uses a different example from geometry. How do we prove that the angle sum in a triangle amounts to two right angles? The philosopher would try, Kant writes, to analyze the concept of triangle, but:

"(…) he may analyze the conception of a straight line, of an angle, or of the number three as long as he pleases, but he will not discover any properties not contained in these conceptions. However, if this question is proposed to a geometrician, he at once begins by constructing a triangle. He knows that two right angles are equal to the sum of all the contiguous angles, which proceed from one point in a straight line; and he goes on to produce one side of his triangle, thus forming two adjacent angles which are together equal to two right angles ... ". (Kant, B 744).

But sometimes in mathematics like in philosophy we employ real conceptual thinking particularly so because calculations or constructions become too intricate end complex.

Again we shall give some examples.

In Otte (2006) we have extensively discussed the following:

Theorem: The orthocenter O, centroid CG and circumcenter M of any triangle are collinear. The line passing through these points is called the Euler line of the triangle. The centroid divides the distance from the orthocenter to the circumcenter in the ratio 2:1.

By analyzing the proofs of this theorem as presented by textbooks of elementary geometry, one might hit upon the idea that the theorem is not at all about the relations between different properties of a single triangle. But rather is an affirmation about the relation between on and the same property (namely the location of the orthocenter) of two different triangles (the original one and its medial triangle, the triangle formed by joining the midpoints of the sides of the given triangle). (Fig.7, p. 148).

Now these two triangles are related to one another by means of a rotation of 180 degrees about the centroid of the given triangle and an additional shrinking of the rotated triangle towards the centroid to half its size. Thus the image point X' of any point X of the plane under this transformation lies on the line that contains X and the centroid, the center of the transformation, to the other side of the centroid and half the distance from it. This completes the proof.

Look however on an Euclidean proof (the same type Kant had in mind) in Otte (2006), Fig 8, p. 149.

Now to give an example from algebra to see how complicated and involved calculations may become take the theorem about the determinant of the product of two real n x n matrices *A* and *B*, which stablishes

$$det(A.B) = det(A) \cdot det(B)$$

The proof in the general case becomes so complicated as

to contain n-times summations in the most intricate way (see appendix).

However, before we come to this theorem we should like to see how the fact that mathematical concepts either may serve as nouns or as rules of inferences, like in the case of the odd number or in Kant's example is treated by linguistics.

2. Semiotics and Complementarity

We do encounter the same type of complementarity with respect to linguistic behavior.

Jakobson, for example, classified linguistic behavior as referring to either code or context, and has accordingly described the diverse forms of aphasia in relation to disturbances of either of these references. For aphasics of the first type context is the indispensable and decisive factor. Their behavior is characterized by Jakobson as "a loss of meta-language" rendering them incapable of uttering a predication that has not been stimulated by the context at hand.

In the pathological cases under discussion, an isolated word means actually nothing but blab. As numerous tests have disclosed, for such patients two occurrences of the same word in two different contexts are mere homonyms. Such a person may never utter the word *knife* alone, but, according to its use and surroundings.

"alternately call the knife *pencil-sharpener, apple-parer, bread-knife, ...* so that the word knife was changed from a free form, capable of occurring alone, into a bound form... . The patient was able to select the appropriate term *bachelor* when it was supported by the context ... but was incapable of utilizing the substitution set *bachelor = unmarried man* as the topic of a sentence, because the ability for autonomous selection and substitution had been affected. [These patients cannot] be brought to understand the metaphoric use of words" (Jakobson, 1956, p. 79–80).

In the second type of aphasia, described by Jakobson as *contiguity disorder*, that is, the ability to construct contexts is impaired. The syntactical rules of organizing words into higher units are lost....

"This type of aphasia tends to give rise to infantile one-sentence utterances and one-word sentences. The patient confined to the substitution set deals with similarities, and his approximate identifications are of a metaphoric nature, contrary to the metonymic ones familiar to the opposite type of aphasics". (ibid., p. 85–86)

One could call one type of aphasia a loss of predication, or, using semiotic terminology, a lack of iconicity and the other a loss of instrumental or functional orientation. The less a word depends grammatically on the context, the stronger is its tenacity in the speech of aphasics with a contiguity disorder and the earlier it is dropped by patients with a similarity disorder. (ibid., p. 86)

Mathematics, considered as semiotic activity, is to be characterized by the related complementarity, or saying it differently, by the necessity of establishing such a complementarity within the process and evolution of cognitive activity.

3. A Theorem about Determinants

Theorem: Let A and B be n x n matrices with real coefficients. Then det(A.B) = det(A) . det(B) .

Now, first, what kind of object is a matrix? A matrix is nothing but a scheme of numbers and it can mean many things: linear transformations, tensors, production-schemes or whatever.

As we have said, we would like to avoid that frightening calculations in the general case. Our strategy will be the following:

1. We treat the simple case of the determinant of 2x2 matrices. In this case, the theorem is verified easily by calculation.
2. We interpret the modulus of a determinant in terms of plane geometry as the area of a parallelogram, where the rows of the matrix are occupied by the coordinates of the sides of the parallelogram.
3. To do that we begin with the simplest case of a rectangle, whose sides are on the coordinate axes and then we show how the area remains invariant under certain transformations. For example if we add a multiple of one side of the parallelogram to the other side (in terms of vector addition) the area does not change.
4. We verify that the corresponding algebraic transformations of the matrix - that is adding a multiple of one row to the other row does not change the determinant. This concludes the proof that the area of the parallelogram can be seen as the modulus of the vector product of its sides generalizing the familiar case of the rectangle area.
5. The theorem *det(A.B) = det(A) . det(B)* can be interpreted geometrically as follows. We consider *C = A.B* as a parallelogram, which is an image of the parallelogram B after a linear transformation represented by the matrix A is applied to it. If | *det(A)* |= *1* than the area remains invariant. This means that the set of areas or the moduli of vector products forms a set which is isomorphic to the quotient group Gl(2)/Sl(2), where Gl(2) is the group of invertible linear transformations and Sl(2) is the subgroup of transformations or matrixes with determinant equal to +1 or -1. Now Gl(2)/Sl(2) is a one-dimensional vector-space. In fact, it is nothing but the set of multiples of the unit matrix.

We can reformulate our result in the following way: The set of area functions, that is, the set of bilinear an anti-symmetric functions f(x,y), where x and y are two-dimensional vectors, is itself a one-dimensional vector-space.

Now we generalize this last result to the n-dimensional

case. So let $\Phi(x_1, \dots x_n)$ be a determinant function, that is, it is a multi-linear and anti-symmetric function on the vector-space V and let $(a_1, \dots a_n)$ be a basis of this vector space such that $\Phi(a_1, \dots a_n) = 1$.

Finally let $\Psi(x_1, \dots x_n)$ be a second determinant function, then we have

$$\Psi(x_1, \dots x_n) = r. \Phi(x_1, \dots x_n) \quad (**),$$

where r is a real number.

To prove (**) we need only define r as equal to $\Psi(a_1, \dots a_n)$. Then $\Psi(x_1, \dots x_n)$ and $\Phi(x_1, \dots x_n)$ being identical on a basis $(a_1, \dots a_n)$ are also identical on the whole vector-space V.

Appendix

Theorem. Let A and B be matrices of order n over the real field.

Then $\det(AB) = \det(A) . \det(B)$.

Proof: Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = AB = (c_{ij})$. Then

$$Cij = \sum_{k=1}^{n} a_{ik} b_{kj} \quad (i, j = 1, \dots, n).$$

So $\det(C) = det\begin{pmatrix} \sum a_{1k_1}b_{k_11} & \sum a_{1k_2}b_{k_22} & \sum a_{1k_n}b_{k_nn} \\ \cdots & \cdots & \cdots \\ \sum a_{nk_1}b_{k_11} & \sum a_{nk_2}b_{k_22} & \sum a_{nk_n}b_{k_nn} \end{pmatrix}$

$$\overset{(1)}{=} \sum_{k_1}\sum_{k_2}\cdots\sum_{k_n} det\begin{pmatrix} a_{1k_1}b_{k_11} & a_{1k_2}b_{k_22} & \cdots & a_{1k_n}b_{k_nn} \\ \cdots & \cdots & & \cdots \\ a_{nk_1}b_{k_11} & a_{nk_2}b_{k_22} & \cdots & a_{nk_n}b_{k_nn} \end{pmatrix}$$

$$\overset{(2)}{=} \sum_{(k_1,\dots,k_n)} b_{k_11}b_{k_22}\dots b_{k_nn}\, det\begin{pmatrix} a_{1k_1} & a_{1k_2} & \cdots & a_{1k_n} \\ \cdots & & \cdots & \cdots \\ a_{nk_1} & a_{nk_2} & \cdots & a_{nk_n} \end{pmatrix}$$

$$\overset{(3)}{=} \sum_{\substack{(k_1,\dots,k_n) \\ k_i \neq k_j}} b_{k_11}b_{k_22}\dots b_{k_nn}\, det\begin{pmatrix} a_{1k_1} & \cdots & a_{1k_n} \\ \cdots & \cdots & \cdots \\ a_{nk_1} & \cdots & a_{nk_n} \end{pmatrix}$$

$$\overset{(4)}{=} \sum_{\sigma} b_{k_11}b_{k_22}\dots b_{k_nn}\, sgn(\sigma)\det(A)$$

$$\overset{(5)}{=} \det(A) \sum_{\sigma} sgn(\sigma)b_{k_11}b_{k_22}\dots b_{k_nn}$$

$$\overset{(6)}{=} \det(A) \sum_{\sigma} sgn(\sigma)b_{1k_1}b_{2k_2}\dots b_{nk_n} \overset{(7)}{=} \det(A).\det(B)$$

Explanations:

1. The determinant is a linear function in each column.
2. Linearity of the determinant again
3. Elimination of the parcels in which $k_i = k_j$ com $i \neq j$ since, in this case

$$det\begin{pmatrix} a_{1k_1} & \cdots & a_{1k_n} \\ \cdots & \cdots & \cdots \\ a_{nk_1} & \cdots & a_{nk_n} \end{pmatrix} = 0$$

4. With the hypothesis that $k_i \neq k_j$ for $i \neq j$, the matrices

$$\begin{pmatrix} a_{1k_1} & \cdots & a_{1k_n} \\ \cdots & \cdots & \cdots \\ a_{nk_1} & \cdots & a_{nk_n} \end{pmatrix}$$

have the same columns as the matrix A with a permutation σ. If

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$$

then the determinant of that matrix is equal to the determinant of A multiplied by $sgn(\sigma)$, the signal of σ.

5. Obvious
6. A permutation and its inverse have the same signal.
7. Definition of determinant of a matrix.

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