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Generalizations of Pythagoras' theorem

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Abstract: Pythagoras' Theorem is one of the most fascinating results in the History of Mathematics. Although there are indications that the result was already known before by the Babylonians, was with the Pythagorean School that there was a formal demonstration of this theorem. As Loomis (1972), in 1940 were known at least 340 different demonstrations of the Pythagoras' Theorem, whose enunciation is as follows: "In any right triangle, the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares whose sides are over each cathetus". This article discusses Pythagoras' Theorem and some generalizations, and introduces a new generalization of this important theorem.

Keywords: Mathematics Education, Geometry, Pythagoras' Theorem

1. Introduction

Pythagoras was born on the island of Samos, in Asia Minor, around the year 569 BC. He went to Egypt and in Babylonia where he could absorb the mathematical knowledge and local religious ideas. Returning to the Greek world, founded a school dedicated to the study of mathematics and philosophy, the *Pythagorean School*.

One of the major contributions of the *Pythagorean School* was organizing parts of the knowledge of geometry through theorems. As no written of the *Pythagorean School* exists today, the information from this period are derived from indirect sources.

Pythagoras' Theorem is one of the most fascinating results in the History of Mathematics. Although there are indications that the result was already known before by the Babylonians, was with the Pythagorean School that there was a formal demonstration of this theorem. As Loomis (1972), in 1940 were known at least 340 different demonstrations of the Pythagoras' Theorem, whose enunciation is as follows:

> *"In any right triangle, the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares whose sides are over each cathetus".*

2. Generalizations of Pythagoras' Theorem

In this section, we expose some generalizations of this theorem and present yet another generalization.

During this section will work with the following basic construction:

Let ABC be a triangle with sides a = BC, b = AC and c = AB. By A one traces a perpendicular line to the side BC that intercepts it in the point H. One considers now the rectangle triangle AHC and AHB with hypotenuses b = AC and c = AB, respectively, and one calls h = AH, m = CH and n = HB.

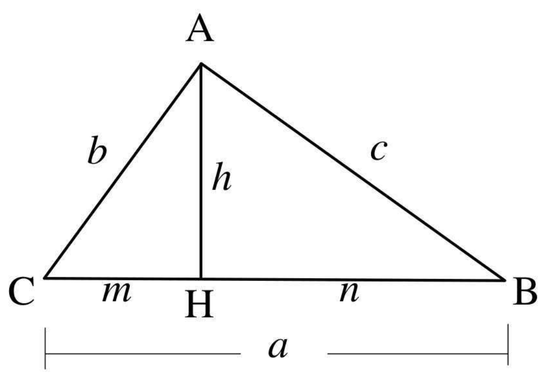

Figure 1. *The basic construction.*

In relation to the basic construction, *Pythagoras' Theorem* and its reciprocal say the following:

Theorem: The triangle ABC is rectangle (in A) if and only if $a^2 = b^2 + c^2$, that is, the triangle ABC is rectangle if and only if the sum of the areas of the squares built over the cathetus b and c is equal to the area of the square built over the hypotenuse a.

We can have a visualization of this theorem in the following picture:

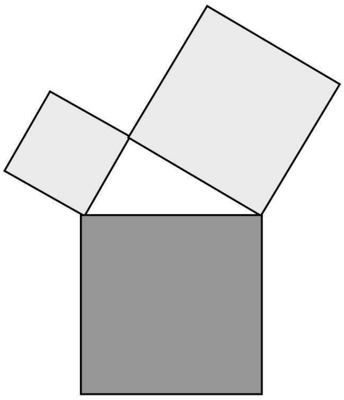

Figure 2. Geometric visualization of Phytaghoras' Theorem.

A generalization of *Phytagoras' Theorem*, known since the Antiquity, changes the pictures of the squares by similar regions built over the sides of the triangle and says:

Theorem: Given a triangle with sides a, b and c, let A, B and C be the areas of similar pictures built over the sides of the triangle. Then, the triangle is rectangle if and only if A = B + C, that is, the triangle is rectangle if and only if the sum of the areas of the similar pictures built over the sides b and c is equal to the area of the similar picture built over the side a.

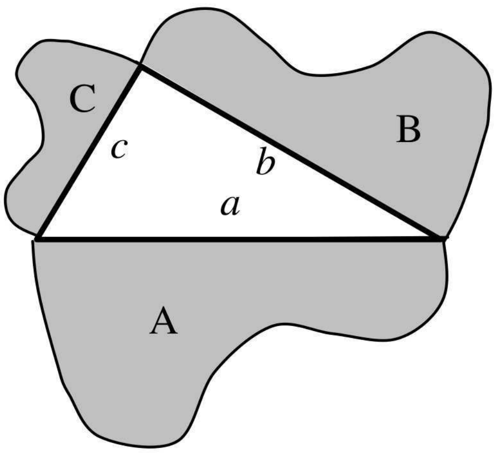

Figure 3. Geometric visualization of the general Pythagoras' Theorem.

Taking an algebraic view over the problem, we realize that the similar pictures can be substituted by any pictures which areas are proportional to the square of the respective sides. Then we propose:

Theorem: Given the triangle ABC according to the basic construction, let $\alpha, \beta \text{ e } \gamma$ be the areas of the pictures associated to the sides a, b and c, respectively, and such that $\frac{\alpha}{a^2} = \frac{\beta}{b^2} = \frac{\gamma}{c^2}$. Then the triangle ABC is rectangle (in A) if and only if $\alpha = \beta + \gamma$, that is, the triangle ABC is rectangle if and only if the sum of the areas of the pictures associated to the sides b and c is equal to the area of the picture associated to the side a.

In fact, this theorem is a particular case of the following algebraic result:

Theorem: One considers the basic construction. Suppose that the triangle ABC is rectangle, and, as before, one considers the rectangle triangles AHC and AHB. Let $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ be the areas of the triangles ABC, AHC and AHB, respectively. Let now α, β and γ be real numbers with $\alpha \cdot \beta \cdot \gamma \neq 0$ and such that $\frac{\alpha}{a^2} = \frac{\beta}{b^2} = \frac{\gamma}{c^2}$. Then

$$\frac{\alpha}{\mathcal{A}} = \frac{\beta}{\mathcal{A}_1} = \frac{\gamma}{\mathcal{A}_2} \text{ and } \alpha = \beta + \gamma.$$

Proof: The triangle ABC is similar to the triangle AHC, and from this similarity one has $\frac{a}{b} = \frac{b}{m} = \frac{c}{h}$, which implies that $b^2 = a.m$.

The triangle ABC is similar to the triangle AHB, and from this similarity one has $\frac{a}{c} = \frac{b}{h} = \frac{c}{n}$, which implies that $c^2 = a.n$.

Let $\mathcal{A}$, $\mathcal{A}_1$ e $\mathcal{A}_2$ be the areas of the triangles ABC, AHC e AHB respectively.

Then, $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$.

Calculating the areas, one has

$$\mathcal{A} = \frac{ah}{2},\ \mathcal{A}_1 = \frac{mh}{2} \text{ and } \mathcal{A}_2 = \frac{nh}{2}.$$

$$\text{Then } \frac{\mathcal{A}}{\mathcal{A}_1} = \frac{\frac{1}{2}.ah}{\frac{1}{2}.mh} = \frac{a}{m} = \frac{a.a}{m.a} = \frac{a^2}{b^2} \text{ or } \frac{\mathcal{A}}{a^2} = \frac{\mathcal{A}_1}{b^2},$$

$$\frac{\mathcal{A}}{\mathcal{A}_2} = \frac{\frac{1}{2}.ah}{\frac{1}{2}.nh} = \frac{a}{n} = \frac{a.a}{n.a} = \frac{a^2}{c^2} \text{ or } \frac{\mathcal{A}}{a^2} = \frac{\mathcal{A}_2}{c^2}$$

It follows that $\frac{\mathcal{A}}{a^2} = \frac{\mathcal{A}_1}{b^2} = \frac{\mathcal{A}_2}{c^2}$ and, since $\frac{\alpha}{a^2} = \frac{\beta}{b^2} = \frac{\gamma}{c^2}$, one has that $\frac{\alpha}{\mathcal{A}} = \frac{\beta}{\mathcal{A}_1} = \frac{\gamma}{\mathcal{A}_2}$.

Since $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, one concludes that α = β + γ.

Corollary: (Pythagoras' Theorem) Taking $\alpha = a^2$, $\beta = b^2$ e $\gamma = c^2$ in the precedent theorem, it follows that $a^2 = b^2 + c^2$.

We joint the results in the following:

Theorem: Given the triangle ABC as in the basic construction, let α, β e γ be real numbers with $\alpha \cdot \beta \cdot \gamma \neq 0$ and such that $\frac{\alpha}{a^2} = \frac{\beta}{b^2} = \frac{\gamma}{c^2}$. The following affirmations are equivalent:

The triangle ABC is rectangle.

1. $\alpha = \beta + \gamma$
2. $a^2 = b^2 + c^2$

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